USING APPROXIMATE ANALYTIC METHODS TO SOLVE PROBLEMS OF NATURAL CONVECTION IN A CLOSED CAVITY

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A mathematical analogy between plate deflection under a transverse load and the circulatory motion of a liquid in a geometrically similar region is validated. A Galerkin-method approximate analytic solution is given for the stream function in a cylindrical cavity.

Problems involving natural convection in closed cavities are very difficult to solve by analytic methods at present. The circulatory motion produced by natural convection in the liquid within the cavity is described by nonlinear equations, so that rigorous analytic solution of the problem is only possible in rare cases, with fairly coarse assumptions being made. Thus numerical methods have usually been employed for solution of such problems in the complete formulation in recent studies. With all their indisputable advantages, numerical methods, which yield extremely rich information about the parameters of a physical process, still involve certain difficulties associated with approximation errors introduced by difference schemes and with problems of stability and convergence of such schemes. In solving such problems, therefore, it is evidently desirable to consider analytic methods of investigation in addition to numerical techniques.

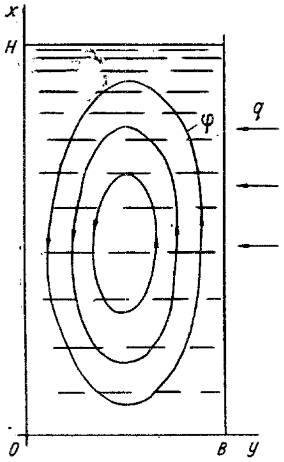

Fig. 1

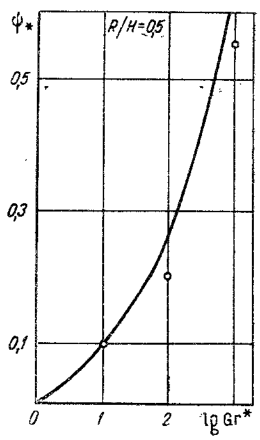

Fig. 2

Fig. 1. Qualitative streamline pattern in vessel for small Grashof numbers, quasi-stationary conditions.

Fig. 2. Variation in extremum value of stream function: the solid line represents the analytic solution; the dots represent the numerical solution.

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To bring out the essence of our approach, let us first look at the problem of circulatory motion of a liquid in a plane rectangular cavity. The liquid is caused to move by the natural convection which arises with a change in the thermal conditions at the side walls of the cavity. No heat is added or subtracted at the free surface of the liquid or at the bottom, while thermal boundary conditions of the second kind are prescribed at the side walls; the specific heat flux q at these walls is assumed to be constant in time.

Under quasi-stationary conditions, when the velocity field in the liquid does not vary in time, the equation for the stream function that characterizes the intensity of motion of the viscous-incompressible liquid has the form

$$\frac{\partial^4 \varphi}{\partial x^4} + 2\frac{\partial^4 \varphi}{\partial x^2 \partial y^2} + \frac{\partial^4 \varphi}{\partial y^2} = -\frac{g\beta}{\nu}\frac{\partial T}{\partial y} - \frac{1}{\nu}\left(\frac{\partial \varphi}{\partial x}\cdot\frac{\partial}{\partial y} - \frac{\partial \varphi}{\partial y}\cdot\frac{\partial}{\partial x}\right)\left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}\right) \tag{1}$$

with the following boundary conditions:

$$\begin{aligned} \text{for } x = 0 \quad & \varphi = 0, \quad \partial\varphi/\partial x = 0; \\ x = H \quad & \varphi = 0, \quad \partial^2\varphi/\partial x^2 = 0; \\ y = 0 \quad & \varphi = 0, \quad \partial\varphi/\partial y = 0; \\ y = B \quad & \varphi = 0, \quad \partial\varphi/\partial y = 0. \end{aligned} \tag{2}$$

When the liquid is heated through the side wall, circulatory motion appears in the cavity with the particles of liquid moving upward at the heated wall and descending at the center of the cavity. Figure 1 qualitatively illustrates the streamline pattern in the symmetric portion of the cavity.

If the streamlines of Fig. 1 are interpreted as geodesic lines, then the profile of the stream function φ will be analogous to the profile of a geometrically similar plate deflected by a transverse load. For the given case this analogy has a quite rigorous mathematical foundation [1], since the equation for the deflection of a rectangular plate under a transverse load has the same form as (1); the analog of the kinematic-viscosity coefficient φ in our case is the cylindrical rigidity of the plate. The boundary conditions for the stream function φ [Eqs. (2)] may also be interpreted as the conditions for rigid clamping and hinge support of the plate [1].

For the axisymmetric region corresponding to a cavity in a cylindrical vessel there is also an analogy between the profile of the stream function ψ and the deflection of a rectangular plate, but here the plate must be treated as having variable flexural rigidity, i.e., the plate thickness may vary in the direction corresponding to the radial direction in the cylindrical cavity.

For quasi-stationary conditions, the circulatory motion of the liquid in a cylindrical vessel will be described by a differential equation form,

$$\begin{aligned} L(\bar\psi) = {} & \frac{1}{\bar r}\left[\frac{\partial\bar\psi}{\partial\bar z}\left(\frac{\partial}{\partial\bar r} - \frac{1}{\bar r}\right) - \frac{\partial\bar\psi}{\partial\bar r}\frac{\partial}{\partial\bar z}\right]\left[\frac{1}{\bar r}\left(\frac{R}{H}\right)^2\frac{\partial^2\bar\psi}{\partial\bar z^2}\right. \\ & \left. + \frac{\partial}{\partial\bar r}\left(\frac{1}{\bar r}\frac{\partial\bar\psi}{\partial\bar r}\right)\right] + \left(\frac{R}{H}\right)^2\frac{\partial^2}{\partial\bar z^2}\left[\frac{1}{\bar r}\frac{\partial^2\bar\psi}{\partial\bar z^2}\left(\frac{R}{H}\right)^2 + \frac{\partial}{\partial\bar r}\left(\frac{1}{\bar r}\frac{\partial\bar\psi}{\partial\bar r}\right)\right] \\ & + \frac{1}{\bar r}\left(\frac{R}{H}\right)^2\frac{\partial}{\partial\bar r}\left[\bar r\frac{\partial}{\partial\bar r}\left(\frac{1}{\bar r}\frac{\partial^2\bar\psi}{\partial\bar z^2}\right)\right] \\ & + \frac{\partial}{\partial\bar r}\frac{1}{\bar r}\frac{\partial}{\partial\bar r}\bar r\frac{\partial}{\partial\bar r}\left(\frac{1}{\bar r}\frac{\partial\bar\psi}{\partial\bar r}\right) + \frac{R}{H}\mathrm{Gr}^*\frac{\partial\theta}{\partial\bar r} = 0, \end{aligned} \tag{3}$$

where the term containing the radial temperature gradient will also be time-independent by virtue of the thermal quasi-stationarity. The function ψ should satisfy boundary conditions of the form

$$\begin{aligned} \text{for } \bar z = 0 \quad & \bar\psi = 0, \quad \partial\bar\psi/\partial z = 0; \\ \bar z = 1 \quad & \bar\psi = 0, \quad \partial^2\bar\psi/\partial\bar z^2 = 0; \\ \bar r = 0 \quad & \bar\psi = 0, \quad \partial\bar\psi/\partial\bar r = 0; \\ \bar r = 1 \quad & \bar\psi = 0, \quad \partial\bar\psi/\partial\bar r = 0. \end{aligned} \tag{4}$$

On the basis of the analogy with plate deflection we use the analytic Galerkin method, which has been quite well worked out for problems of structural mechanics, to solve (3) with the boundary conditions (4). We specify a function of two variables in the form

$$\bar{\psi} = Af(\bar{z}, \bar{r}) = A\bar{z}^2(3 - 5\bar{z} + 2\bar{z}^2)\bar{r}^2(1 - \bar{r})^2, \tag{5}$$

that will satisfy the boundary conditions in the form (4) while qualitatively reflecting the nature of the circulatory motion of the liquid in the vessel. The solution now reduces to determination of the coefficient A. To do this, we use the Galerkin-method procedure

$$\int_0^1\int_0^1 L[Af(\bar{z}, \bar{r})]f(\bar{z}, \bar{r})\,\bar{r}d\bar{r}d\bar{z} = 0. \tag{6}$$

Taking the dependence corresponding to quasi-stationary thermal conditions in the motionless liquid as the first approximation of the radial temperature gradient, we obtain

$$A = -77G\left(\sqrt{1 + \frac{H}{R}\frac{\mathrm{Gr}^*}{77G^2}} - 1\right), \tag{7}$$

after certain calculations, where

$$G = \frac{1}{2}\left(\frac{H}{R}\right)^2\left[\frac{95}{9} + \frac{64}{7}\left(\frac{R}{H}\right)^2 + 8\left(\frac{R}{H}\right)^4\right].$$

We note that an extremum of the stream function $\bar{\psi}$, taken in the form (5), is reached at $\bar{z} = 0.578$, $\bar{r} = 0.5$ and equals $\psi^* = 0.0163\,A$. The approximate analytic solution yields a dependence for the extremum value of the stream function ψ_* that is in fairly good agreement with the results of the numerical solution (Fig. 2) for circulatory motion of fairly low intensity (up to $\mathrm{Gr}^* = 10^3$); here the symmetry of the pattern of stream-function distribution is still preserved, and there is little shift in its extremum with respect to the center of the given cavity cross section. As Gr* increases, the symmetry of motion is destroyed, the center of the vortex shifts, the streamline is distorted, and there are secondary vortices with reverse motion. In this case the mathematical analogy with the deflection plate loses its clarity and becomes difficult to use. Then the basic means of investigating the process becomes the numerical solution of the given problem.

In particular, the numerical results of Fig. 2 were obtained by integrating a system of partial differential equations of the parabolic type in the given region; these equations characterize the transfer of energy and momentum in conjunction with the elliptic differential equation for the stream function. Such a mathematical description of the natural-convection process in a vertical-axisymmetric vessel is equivalent to a description by (3) in operator form. The numerical solution was obtained for a grid region containing 41×41 node points; an adjustment method was used with the transfer equations being integrated by means of an explicit finite-difference scheme with the convective terms of the equations being approximated by differences oriented against the flow. The elliptic equation was integrated by an implicit finite-difference scheme in conjunction with the procedure of scalar dispersion with respect to variable directions.

To conclude, we note that the approximate analytic solution obtained for the stream function with a natural-convection process of low intensity enables us to evaluate the temperature and concentration distributions of matter within the liquid volume for complex heat and mass-transfer processes in the presence of natural convection (for example, dissolution of gases in a liquid partially filling a cylindrical vessel) without going through the laborious calculations of the velocity fields which require much machine time.

NOTATION

β is the coefficient of bulk thermal expansion of the liquid; q is the specific heat flux at the vessel wall; x, y and z, r are the coordinates in the Cartesian and cylindrical systems; φ and ψ are the stream functions in rectangular and cylindrical cavities; H is the height of the liquid level in the cavity; B is half the width of the rectangular cavity; R is the radius of the cylindrical cavity; T is the temperature; g is the acceleration of the mass-force field; ν is the kinematic-viscosity coefficient; λ is the thermal-conductivity coefficient of the liquid; A is the coefficient in Eq. (5) for the stream function; G is a coefficient in (7) that depends on the geometry of the cylindrical cavity; L( ) is a differential operator; we have the following dimensionless parameters: $\bar{\psi} = \psi/(\nu H)$, $\theta = T/(qR/\lambda)$, $\bar{z} = z/H$, $\bar{r} = r/R$; $\mathrm{Gr}^* = g\beta(qR/\lambda)R^3/\nu^2$ is a modified Grashof number.

LITERATURE CITED

1. V. S. Kuptsova, "On the mathematical analogy between bending of a plate and the circulatory motion of a liquid in a geometrically similar region," Izv. Vyssh. Uchebn. Zaved., Mashinostroenie, No. 11, 72 (1972).